An Application of Group Theory to Matrices and to Ordinary Differential Equations*

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ABSTRACT

A result concerned with groups is proved, from which several applications can be derived. We estimate e.g. the number of distinct eigenvalues of the Kronecker product and sum of two given matrices A, B, when A as well as B has distinct eigenvalues. We also discuss the order of the linear ODE whose solutions are the products of solutions of two given linear ODEs, when such ODEs are in certain classes.

1. A RESULT OF GROUP THEORY

In this section we prove a simple theorem concerned with groups, from which we shall derive several applications. In the next section we shall apply this result, in particular, to matrix theory and to ordinary differential equations (DEs).

More precisely, we shall obtain an estimate for the number of distinct eigenvalues of the Kronecker product and sum of two given matrices A, B, when A as well as B has distinct eigenvalues. Further, drawing on the same arguments, we shall discuss the minimum order μ of the linear, homogeneous, ordinary DE whose solutions are the products of solutions of two given DEs, when such DEs have constant coefficients, and in some related cases. Finally some estimates for μ, useful in the theory of *special functions*, are derived.

Below the additive notation is adopted, as is customary for abelian groups.

THEOREM 1.1. *Let $\mathcal{G}$ be an abelian group of order $|\mathcal{G}|\leqslant\infty$; $\mathcal{A}=\{\alpha_i\}$, $\mathcal{B}=\{\beta_j\}$ two nonempty finite subsets of $\mathcal{G}$; $|\mathcal{A}|$ the number of elements of $\mathcal{A}$;*

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$\mathfrak{A} + \mathfrak{B} = \{\alpha_i + \beta_j\}$. *Suppose* $|\mathfrak{A}| \geqslant |\mathfrak{B}| > k \geqslant 1$, k *integer. If there is an element* $\beta_{j'}$ *in* $\mathfrak{B}$ *such that the relation*

$$(|\mathfrak{A}| + h)(\beta - \beta_{j'}) \neq 0 \tag{1.1}$$

has not less *than* $h+1$ *solutions* β *in* $\mathfrak{B}$ *for* $h = 0, 1, \ldots, |\mathfrak{B}| - k - 1$, *then*

$$|\mathfrak{A} + \mathfrak{B}| \geqslant |\mathfrak{A}| + |\mathfrak{B}| - k. \tag{1.2}$$

If $|\mathfrak{A}| + |\mathfrak{B}| > |\mathcal{G}|$, then $\mathfrak{A} + \mathfrak{B} = \mathcal{G}$. We assume $\mathcal{G} \neq \mathfrak{A} + \mathfrak{B}$ and therefore $|\mathfrak{A}| + |\mathfrak{B}| \leqslant |\mathcal{G}|$ (see [6, p. 1]).

Recall that (1.2) holds, with $k = 1$, (a) if $\mathcal{G}$ is *cyclic* of prime order (Cauchy and Davenport; see e.g. [6, p. 3]), and (b) if $\mathcal{G}$ is the *cyclic group of residue classes* mod $|\mathcal{G}|$, with $|\mathcal{G}|$ composite, $\beta_1 = 0$, and $\beta_2, \beta_3, \ldots, \beta_n$ prime to $|\mathcal{G}|$ (Chowla; see [6, p. 3]).

Moreover, Scherk and Kemperman [10] proved (1.2) for arbitrary abelian groups, subject to some additional hypotheses.

Proof. The proof is by induction on $|\mathfrak{B}| =: n$. Suppose first $k = 1$.

For $n = 1$ there is nothing to prove, because the elements $\alpha_i + \beta_1$, $i = 1, 2, \ldots, m := |\mathfrak{A}|$, are distinct.

For $n = 2$, let $j' = 1$ in (1.1). Then (1.1) with $h = 0$, i.e. $m(\beta - \beta_1) \neq 0$, has at least one solution in $\mathfrak{B}$, which must coincide with β_2. It follows that $\Sigma_{i=1}^{m}(\alpha_i + \beta_1) \neq \Sigma_{i=1}^{m}(\alpha_i + \beta_2)$ and therefore the distinct elements in $\mathcal{G}$ having the form $\alpha_i + \beta_1, \alpha_i + \beta_2$ are at least $m+1$.

For $n > 2$, suppose that the theorem (with $k = 1$) holds for all $n' < n$. We shall apply this result to the two sets $\{\gamma_1, \gamma_2, \ldots, \gamma_l\}$, $\{\beta_1, \beta_n\}$, where the γ_r's are the distinct elements in $\mathcal{G}$ having the form $\alpha_i + \beta_j$; therefore $l = |\mathfrak{A} + \mathfrak{B}|$. By the inductive hypothesis for $n' = n-1$, we have $l \geqslant m + n - 2$, and if $l = m + n - 2$, we obtain from (1.1) with $h = n-2$ that $l(\beta - \beta_1) \neq 0$ for at least $n-1$ elements β in $\mathfrak{B}$. Therefore necessarily $\beta = \beta_n$ is a solution and so

$$\sum_{r=1}^{m+n-2} (\gamma_r + \beta_1) \neq \sum_{r=1}^{m+n-2} (\gamma_r + \beta_n) \quad \text{and} \quad l \geqslant m + n - 1.$$

If we stop the procedure at the $\bar{n}$th row, where $\bar{n} = n + 1 - k$, we see that $(m + n - k - 1)(\beta - \beta_1) \neq 0$ has at least $n - k$ solutions in $\mathfrak{B}$ and therefore $l \geqslant m + n - k$. This proves the theorem for a generic $k \geqslant 1$. ∎

REMARK 1.2. The estimate (1.2) can also be obtained from Condition Γ_3, p. 232 of [10] (see also Condition Γ_6, p. 237). The method we followed here is similar, in the beginning, to that of Davenport [5].

REMARK 1.3.

(i) In *any* (even nonabelian) group,

$$|\mathcal{A}+\mathcal{B}| \geqslant \max\{|\mathcal{A}|, |\mathcal{B}|\}. \tag{1.3}$$

(ii) If there exist i, j, $i \neq j$, in the set $\{1,2,\dots,n\}$, such that $|\mathcal{A}|\cdot(\beta_i-\beta_j) \neq 0$, then

$$|\mathcal{A}+\mathcal{B}| \geqslant |\mathcal{A}|+1, \tag{1.4}$$

and therefore if there exist also r, s, $r \neq s$, in $\{1,2,\dots,m\}$, such that $|\mathcal{B}|\cdot(\alpha_r - \alpha_s) \neq 0$, then

$$|\mathcal{A}+\mathcal{B}| \geqslant \max\{|\mathcal{A}|, |\mathcal{B}|\}+1. \tag{1.5}$$

These results can be obtained in the course of the proof above.

REMARK 1.4. In the proof of Theorem 1.1 it suffices that all the elements $\alpha_1,\dots,\alpha_m,\beta_1,\dots,\beta_n$ commute. If the α_i's and the β_j's are given, it also suffices to consider *an* abelian group $\mathcal{G}$ containing them (e.g. the group *finitely generated* by them).

COROLLARY 1.5. *If $\mathcal{G}$ is torsion-free, then* (1.2) *holds with* $k=1$.

Proof. In fact

$$z \in \mathcal{G}, \quad z \neq 0 \quad \Rightarrow \quad pz \neq 0 \quad \text{for any } p=1,2,3,\dots \tag{1.6}$$

in this case, and therefore the hypothesis in Theorem 1.1 is verified also for $k=1$. ■

This corollary can be applied to $(\mathbb{Z},+),(\mathbb{R},+),(\mathbb{R}^+,\cdot)$, etc.

REMARK 1.6. In Corollary 1.5, $|\mathcal{G}|=\infty$ and therefore $\mathcal{G} \neq \mathcal{A}+\mathcal{B}$.

In general, if $\mathcal{G}=\mathcal{A}+\mathcal{B}$, then $|\mathcal{G}|=|\mathcal{A}+\mathcal{B}|\leqslant|\mathcal{A}|\cdot|\mathcal{B}|$, so that $\mathcal{G}$ is finite; therefore, if $|\mathcal{G}|>|\mathcal{A}|\cdot|\mathcal{B}|$, then certainly $\mathcal{G}\neq\mathcal{A}+\mathcal{B}$.

EXAMPLES 1.7. Take *any* two nonempty finite sets of distinct elements in: $\mathcal{G}_1=(\mathbb{R},+)$, $\mathcal{G}_2=(\mathbb{C},+)$; $\mathcal{G}_3=(\mathbb{R}\setminus\{0\},\cdot)$, $\mathcal{G}_4=(\mathbb{C}\setminus\{0\},\cdot)$; $\mathcal{G}_5$, the additive group contained in the division ring of the real quaternions; $\mathcal{G}_6$, the additive group of the $m\times n$ matrices over $\mathbb{C}$. Take two finite sets of commuting matrices in the multiplicative group of the nonsingular square matrices of order n, over $\mathbb{C}$.

In $\mathcal{G}_1,\mathcal{G}_2,\mathcal{G}_5,\mathcal{G}_6$, (1.2) holds, with $k=1$, because equation $pz=0$ with $p=1,2,3,\ldots$ implies $z=0$ [see (1.6)]. In $\mathcal{G}_3$ we have some possible exceptional cases, as the equation $z^p=1$ *may* have real solutions z, $z\neq 1$ (when p is even).

In $\mathcal{G}_4$ we *may* have some exceptions in connection with the $p-1$ pth roots of unity other than 1.

More precisely, let $\mathcal{A},\mathcal{B}$ be two sets respectively of m,n distinct elements in $\mathcal{G}_4=(\mathbb{C}\setminus\{0\},\cdot)$, $m\geqslant n>k\geqslant 1$. If there exists $\beta_{j'}\in\mathcal{B}$ such that there are *at least* $h+1$ β's in $\mathcal{B}$ for which $\operatorname{ord}(\beta/\beta_{j'})\nmid m+h$ or $\operatorname{ord}(\beta/\beta_{j'})=\infty$, for $h=0,1,\ldots,n-k-1$, then (1.2) holds.

EXAMPLES 1.8. $\alpha_j=\beta_j=\omega^{j-1}$ $(j=1,2,\ldots,m=n)$, and therefore $\alpha_i\beta_j=\omega^{i+j-2}$, where ω is any mth order root of unity, $\omega\neq 1$. We have only m distinct elements $\alpha_i\beta_j$, and this is indeed the lowest number possible [cf. (1.3)].

Note that in this case $\mathcal{A}=\mathcal{B}=\mathcal{G}$ and hence $\mathcal{A}+\mathcal{B}=\mathcal{G}$; in fact $|\mathcal{A}|+|\mathcal{B}|=2m>|\mathcal{G}|=m$.

More generally, if $\mathcal{A}$ is *any* finite group of order $|\mathcal{A}|$ and $\mathcal{B}\subseteq\mathcal{A}$ is *any* nonempty subset of $\mathcal{A}$, then $\mathcal{A}+\mathcal{B}=\mathcal{A}=:\mathcal{G}$. Therefore $|\mathcal{B}|\leqslant|\mathcal{A}|=|\mathcal{A}+\mathcal{B}|=|\mathcal{G}|=\max\{|\mathcal{A}|,|\mathcal{B}|\}<|\mathcal{A}|+|\mathcal{B}|$ (cf. (1.3) and [6, p. 1]).

EXAMPLE 1.9.

(a) *"Maximal" case*: $|\mathcal{A}+\mathcal{B}|=|\mathcal{A}|\cdot|\mathcal{B}|$, *in* $(\mathbb{Z},+)$. $\alpha_i=i-1$ $(i=1,2,\ldots,m)$, $\beta_j=m(j-1)+1$ $(j=1,2,\ldots,n)$, and hence $\alpha_i+\beta_j=i+m(j-1)$.

(b) *Case* $|\mathcal{A}+\mathcal{B}|=|\mathcal{A}|+|\mathcal{B}|-1$. $\alpha_i=i-1$ $(i=1,2,\ldots,m)$, $\beta_j=j-1$ $(j=1,2,\ldots,n)$, and hence $\alpha_i+\beta_j=i+j-2$.

Note that it does *not* suffice to have two coinciding sets, $m=n$, $\beta_j=\alpha_j$ $(j=1,2,\ldots,m)$, in order to attain the minimum number of distinct elements having the form $\alpha_i+\beta_j$, which is $2m-1$ in this case.

2. APPLICATIONS

A. *Matrix Theory*

The following result of matrix theory can be obtained as an application of Theorem 1.1. For convenience we state it just for matrices over the complex field.

THEOREM 2.1. *Let A be an $m\times m$ matrix over the complex field $\mathbb{C}$, having the distinct eigenvalues $\alpha_1,\alpha_2,\dots,\alpha_m$, and B an $n\times n$ matrix over $\mathbb{C}$, having the distinct eigenvalues $\beta_1,\beta_2,\dots,\beta_n$.*

(i) *Suppose that $m\geqslant n>k\geqslant 1$, k integer and that all $\alpha_i\neq 0$ and all $\beta_j\neq 0$. If there exists a $\beta_{j'}$, $j'\in\{1,2,\dots,n\}$, such that the equation*

$$(\beta_j/\beta_{j'})^{m+h}\neq 1$$

has at least $h+1$ *solutions in the set $\{\beta_1,\beta_2,\dots,\beta_n\}$, for $h=0,1,\dots,n-k-1$, then the Kronecker product $A\otimes B$ has at least $m+n-k$ distinct eigenvalues.*

(ii) *The "Kronecker sum" $A\otimes I_n+I_m\otimes B$, where I_n denotes the $n\times n$ identity matrix, has always at least $m+n-1$ distinct eigenvalues.*

Proof. In fact the eigenvalues of $A\otimes B$ and $A\otimes I_n+I_m\otimes B$ are respectively $\alpha_i\beta_j$, $\alpha_i+\beta_j$ $(i=1,2,\dots,m;\ j=1,2,\dots,n)$. Theorem 1.1 then proves Theorem 2.1 (cf. also $\mathcal{G}_2$ and $\mathcal{G}_4$ in Examples 1.7). ■

REMARK 2.2. Corollary 1.5 suffices to prove Theorem 2.1(ii) when A, B are *real symmetric* or *Hermitian* matrices, and (i) when A, B are also positive definite. In this case $A\otimes B$, and therefore $A\otimes I_n+I_m\otimes B$, are also real symmetric or Hermitian, respectively (see [7, pp. 8–9]).

Recall that the Kronecker (or direct, or tensor) product plays a role in the study of the matrix algebraic equation $AX+XB=C$ and therefore e.g. in stability theory [1; 7; 2, pp. 234–236]. It is also concerned with stochastic matrices and statistics [2, 9].

The Kronecker sum appears in the study of the matrix DE $X'=AX+XB$ [1].

B. *Differential Equations*

Consider the *multiplicative* abelian group $\mathcal{G}$ of all the nonzero elements in the field $\mathcal{F}$ of the meromorphic functions in a disk Ω.

Let $u_i(x)$ $(i=1,2,\dots,m)$, and $v_j(x)$ $(j=1,2,\dots,n)$, be two fundamental systems for the DEs

$$L_m[u] := u^{(m)} + \sum_{i=1}^{m} a_i u^{(m-i)} = 0, \tag{2.1}$$

$$M_n[v] := v^{(n)} + \sum_{j=1}^{n} b_j v^{(n-j)} = 0, \tag{2.2}$$

respectively, where a_i $(i=1,2,\dots,m)$, b_j $(j=1,2,\dots,n)$ are complex *constants*. Then, in particular, the u_i's are distinct and $u_i \neq 0$ (the function identically zero in Ω), and the same for the v_j's; $u_i, v_j \in \mathcal{G}$.

Suppose $m \geqslant n > 1$. Since the conditions $(v_i/v_j)^p = 1$ for $i \neq j$, $p = 1,2,3,\dots,$ entail linear dependence between v_i, v_j, against assumption, Theorem 1.1 with $k=1$ can be applied to state that *at least* $m+n-1$ *distinct* functions $u_i(x)v_j(x)$ exist.

For the vector space of the solutions of the DEs with constant coefficients, as in (2.1), (2.2), a basis can be chosen by taking some functions of the form $x^p e^{\alpha x}$ $(p \in \mathbb{N}_0,\ \alpha \in \mathbb{C})$. Now, q of these functions are distinct in every $\Omega' \subseteq \Omega$ iff they are linearly independent there, and therefore iff their Wronskian determinant is different from zero in Ω.

Therefore we have proved that the *minimum* order of the linear, homogeneous, ordinary DE with constant coefficients, whose solutions are the products of solutions of (2.1), (2.2), is $m+n-1$ (cf. [4]).

The *actual* order N $(m+n-1 \leqslant N \leqslant mn)$ of the DE "for the products," for two *given* DEs, can be evaluated as the rank of the $mn \times mn$ Wronskian matrix W of the functions $u_i v_j$ $(i=1,2,\dots,m;\ j=1,2,\dots,n)$ at a point $x_0 \in \Omega$. When the roots of the characteristic equations associated with such DEs are all distinct, we have

$$N = \operatorname{rank} W = \operatorname{rank} V(\alpha_1+\beta_1, \alpha_1+\beta_2, \dots, \alpha_m+\beta_n), \tag{2.3}$$

where $V(\alpha_1+\beta_1, \alpha_1+\beta_2, \dots, \alpha_m+\beta_n)$ is the *Vandermonde matrix* of the mn numbers $\alpha_i+\beta_j$. In fact N is the number of *distinct* elements $\alpha_i+\beta_j$ $(i=1,2,\dots,m;\ j=1,2,\dots,n)$.

EXAMPLE 2.3. It is easy to produce various examples for the "maximal" and the "minimal" cases for the DEs as in (2.1), (2.2). By drawing e.g. on

Example 1.9, we obtain $[e^{\alpha_i x}]_{i=1,2,\ldots,m}$, $[e^{\beta_j x}]_{j=1,2,\ldots,n}$ with the same α_i's and β_j's chosen in (a), (b) there.

REMARK 2.4. The estimate $N \geqslant m+n-1$ can be deduced immediately from the corresponding result holding for DEs with constant coefficients, in some other cases—for example, for DEs which can be taken into DEs with constant coefficients under the (global) transformation

$$\tilde{u}(t)=f(t)Au[\phi(t)], \qquad \tilde{v}(t)=g(t)Bv[\phi(t)]. \tag{2.4}$$

Here $u^T := [u_1 \ u_2 \ \cdots \ u_m]$, $v^T := [v_1 \ v_2 \ \cdots \ v_n]$, where T indicates the transpose and u_i $(i=1,2,\ldots,m)$, v_j $(j=1,2,\ldots,n)$ are two fundamental systems for (2.1), (2.2) in the interval I, where a_i, b_j are replaced by conveniently "smooth" functions; f, g are nonvanishing scalar functions $f,g: J \to \mathbb{C}$, J a real interval, $f,g \in C^N(J)$; $\phi(t)$ is a bijection of J onto I, $\phi \in C^N(J)$, $\phi'(t) \neq 0$ on J, and A, B are two nonsingular, constant matrices, whose orders are m, n respectively.

In fact, (2.4) is the most general pointwise transformation which preserves the order and linearity of (2.1), (2.2) (see e.g. [8, p. 310]).

This also happens for the DE for the products. If $w := [w_1 \ w_2 \ \cdots \ w_{mn}]^T$ denotes the column vector obtained by "stacking" the columns Z_k, $k=1,2,\ldots,n$, of the matrix $Z := uv^T$—i.e. $w = \mathcal{S}(uv^T)$, where $\mathcal{S}$ is the linear operator such that $\mathcal{S}(Z)=\mathcal{S}([Z_1 \ Z_2 \ \cdots \ Z_n]) = [Z_1^T \ Z_2^T \ \cdots \ Z_n^T]^T$ (see [2], [9])—then we have

$$\tilde{w} := \mathcal{S}(\tilde{u}\tilde{v}^T) = fg\mathcal{S}(Auv^TB^T) = fg\mathcal{S}(AZB^T) = fg(B \otimes A)w, \tag{2.5}$$

and $\det(fg(B\otimes A)) = (fg)^{mn}(\det A)^n(\det B)^m \neq 0$ in J (and in every $J^* \subset J$).

In particular, as is well known, transforming u as in (2.4) (with $A=I_2$, the 2×2 identity matrix) takes *any* second order DE into *any given* second order DE, e.g. with constant coefficients.

The change of independent variable $x=\log t$, which is also of type (2.4), takes DEs with constant coefficients of any order, with distinct characteristic roots, into the so-called *Euler* DE, and takes the system $[e^{nx}]_{n_1 \leqslant n \leqslant n_2}$, $n_1, n_2, n \in \mathbb{N}_0$, into $[t^n]_{n_1 \leqslant n \leqslant n_2}$.

Much more interesting for the applications is to obtain an estimate for the minimum order μ of the DE "for the products" for more general classes of DEs. Suppose that we are given (2.1), (2.2), with the (conveniently "smooth") coefficients a_i, b_j depending on x in a real bounded open interval I.

PROPOSITION 2.5. *Suppose* $m \geqslant n$; $u_i(x)$ $(i=1,2,\dots,m)$ *is a fundamental system for* (2.1); v_1 *is a solution of* (2.2) *in* I; $u_i, v_1 \in C^m(I)$. *Then* $\mu \geqslant m$ *in some subinterval of* I.

Proof. In fact, as $v_1(x)$ has finitely many zeros in every $[a,b] \subset I$, choose $x_0 \in I$, $v_1(x_0) \neq 0$. Then $v_1(x) \neq 0$ in an interval J, $J \subseteq I$, $x_0 \in J$, and by the "homogeneity property" of the Wronskian determinants, $W(u_1 v_1, u_2 v_1, \dots, u_m v_1) = v_1^m \cdot W(u_1, u_2, \dots, u_m)$, we obtain $W(u_1 v_1, u_2 v_1, \dots, u_m v_1) \neq 0$ in J. Therefore a unique normalized DE can be constructed in J, having continuous coefficients and the system $u_1 v_1, u_2 v_1, \dots, u_m v_1$ as fundamental there (see e.g. [3, pp. 81–84]). ■

This results coincides with that for DEs with constant coefficients when $m \geqslant n = 1$.

PROPOSITION 2.6. *Suppose* $m \geqslant n$; $u_i(x)$ $(i=1,2,\dots,m)$, $v_j(x)$ $(j=1,2,\dots,n)$ *are two fundamental systems for* (2.1), (2.2) *in* I, *respectively, and* $v_1, v_2, u_i \in C^{m+1}(I)$. *Then* $\mu \geqslant m+1$ *in some subinterval of* I.

Proof. As $v_1(x)$ has finitely many zeros in every $[a,b] \subset I$, let $x_0 \in I$ be such that $v_1(x_0) \neq 0$ and therefore $v_1(x) \neq 0$ in an interval $J \subseteq I$, $x_0 \in J$ [alternatively: $W(v_1, v_2, \dots, v_n)(x_0) \neq 0$ implies that $v_j(x_0) \neq 0$ at least for one j]. Setting $u^T := [u_1 \; u_2 \; \cdots \; u_m]$, $v^T := [v_1 \; v_2 \; \cdots \; v_n]$, the m functions $u_i v_1$ $(i=1,2,\dots,m)$ in the first column of matrix uv^T have nonvanishing Wronskian determinant in J (see Proposition 2.5).

Suppose now that every entry in the second column of uv^T is linearly dependent on the $u_i v_1$'s, in a (generic) $J^* \subset J$. Then there exist some complex constants c_{ij}, not all zero, such that

$$u_i v_2 = \sum_{j=1}^{m} c_{ij} u_j v_1 \quad (i=1,2,\dots,m) \qquad \text{in } J^*, \tag{2.6}$$

i.e.

$$(C - \lambda I_m) u = 0 \qquad \text{in } J^*, \tag{2.7}$$

where $C := \{c_{ij}\}$, $\lambda := v_2 / v_1$.

In order to have nontrivial solutions for u, the determinant $\Delta(\lambda) := \det(C - \lambda I_m)$ must vanish in J^*. As $\Delta(\lambda)$ is an mth degree polynomial in λ over $\mathbb{C}$, this entails $\lambda = \text{const}$, against the assumption that v_1, v_2 are linearly independent.

This proves that *at least* $m+1$ linearly independent functions have to appear in uv^T. As J^* is arbitrary and $v_1, v_2, u_i \in C^{m+1}(I)$, we have $W(u_1v_1, u_2v_1, \dots, u_mv_1;\ u_hv_2) \neq 0$ in some $J_1 \subseteq J$ for some h (see [3, pp. 83–84]). ■

Note that this result coincides with that for DEs with constant coefficients when $m \geq n = 2$.

REMARK 2.7. The conclusions above can be extended to *analytic* DEs in $D \setminus \{a\}$, where D is an open disk in the complex plane, around the isolated regular singularity a. The proofs proceed as before, taking $x_0 \in D \setminus \{a\}$ and then extending the results to the whole punctured disk by analytic continuation.

Therefore, in particular, the DEs whose solutions are products of *special functions*, satisfying second order DEs in an annulus around an isolated regular (fuchsian) singularity, may have order 3 *or* 4 (cf. e.g. [11, pp. 147–149] for an application to Bessel functions).

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